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# A Fractional Integral Involving Fractional Trigonometric Function

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***Abstract:*** **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study a fractional integral involving fractional trigonometric function. In fact, our result is a generalization of traditional calculus result.**

***Keywords:*** **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, fractional integral, fractional trigonometric function.**

## I. INTRODUCTION

The history of fractional calculus is almost as long as the development of traditional calculus. In 1695, the concept of fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field, such as Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, and Weyl. In the past few decades, fractional calculus has played a very important role in physics, electrical engineering, economics, biology, control theory, and other fields [1-15].

However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [16-20].

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we study a fractional integral involving fractional trigonometric function. In fact, our result is a generalization of classical calculus result.

## II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

**Definition 2.1** ([21])**:** Let $0 < \alpha \leq 1$, and $x_0$ be a real number. The Jumarie's modified Riemann-Liouville (R-L) $\alpha$-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville $\alpha$-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

**Proposition 2.2** ([22])**:** *If* $\alpha, \beta, x_0, C$ *are real numbers and* $\beta \geq \alpha > 0$, *then*

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$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

*and*

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

**Definition 2.3** ([23])**:** If $x, x_0$, and $a_n$ are real numbers for all $n$, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an $\alpha$-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing $x_0$, then we say that $f_\alpha(x^\alpha)$ is $\alpha$-fractional analytic at $x_0$. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is $\alpha$-fractional analytic at every point in open interval $(a,b)$, then $f_\alpha$ is called an $\alpha$-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

**Definition 2.4** ([24])**:** If $0 < \alpha \le 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two $\alpha$-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

**Definition 2.5** ([25])**:** If $0 < \alpha \le 1$, and $x$ is a real number. The $\alpha$-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{9}$$

On the other hand, the $\alpha$-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{10}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{11}$$

## III. MAIN RESULT

In this section, we solve a fractional integral involving fractional trigonometric function.

**Theorem 3.1:** *Let* $0 < \alpha \le 1$, *then the* $\alpha$*-fractional integral*

$$\left({}_0I_x^{\alpha}\right)\left[\left[1 + sin_\alpha(x^\alpha) \otimes_\alpha cos_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}\right] = \frac{2}{\sqrt{3}} arctan_\alpha\left(\frac{2}{\sqrt{3}}\left(tan_\alpha(x^\alpha) + \frac{1}{2}\right)\right) - \frac{2}{\sqrt{3}} arctan_\alpha\left(\frac{1}{\sqrt{3}}\right). \tag{12}$$

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**Proof** $\left({}_{0}I_{x}^{\alpha}\right)\left[\left[1+sin_{\alpha}(x^{\alpha})\otimes_{\alpha}cos_{\alpha}(x^{\alpha})\right]^{\otimes_{\alpha}(-1)}\right]$

$$=\left({}_{0}I_{x}^{\alpha}\right)\left[\left[\left(sec_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}2}+tan_{\alpha}(x^{\alpha})\right]^{\otimes_{\alpha}(-1)}\otimes_{\alpha}\left(sec_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}2}\right]$$

$$=\left({}_{0}I_{x}^{\alpha}\right)\left[\left[1+\left(tan_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}2}+tan_{\alpha}(x^{\alpha})\right]^{\otimes_{\alpha}(-1)}\otimes_{\alpha}\left({}_{0}D_{x}^{\alpha}\right)\left(tan_{\alpha}(x^{\alpha})\right)\right]$$

$$=\left({}_{0}I_{x}^{\alpha}\right)\left[\left[\left(tan_{\alpha}(x^{\alpha})+\frac{1}{2}\right)^{\otimes_{\alpha}2}+\left(\frac{\sqrt{3}}{2}\right)^{2}\right]^{\otimes_{\alpha}(-1)}\otimes_{\alpha}\left({}_{0}D_{x}^{\alpha}\right)\left(tan_{\alpha}(x^{\alpha})+\frac{1}{2}\right)\right]$$

$$=\frac{2}{\sqrt{3}}arctan_{\alpha}\left(\frac{2}{\sqrt{3}}\left(tan_{\alpha}(x^{\alpha})+\frac{1}{2}\right)\right)-\frac{2}{\sqrt{3}}arctan_{\alpha}\left(\frac{1}{\sqrt{3}}\right).$$

Q.e.d.

## IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we solve a fractional integral involving fractional trigonometric function. Moreover, our result is a generalization of ordinary calculus result. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in engineering mathematics and fractional differential equations.

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